Circle homeomorphisms and shears with ℓ^2 structure

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(joint work with Dragomir Saric and Yilin Wang)

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Massachusetts Institute of Technology

Plan of the talk

Goal: compare circle homeomorphisms defined in terms of shears with ones from Teichmüller theory. Main result:

Theorem (Saric, Wang, W.)

$$\mathcal{C}^{1,\alpha} \subset \mathcal{H} \subset \mathrm{WP}(\mathbb{T}) \subset \mathcal{S}$$

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See preprint at arXiv:2211.11497 for more details and references!

Shears and shear coordinates

Shear in terms of cross ratio

The *cross ratio* of four points a, b, c, d along a circle or line is

$$\mathrm{cr}(a, b, c, d) = \frac{(b-a)(d-c)}{(c-b)(d-a)} \in \mathbb{R}.$$

The cross ratio is invariant under Möbius transformations and has the symmetry $\mathrm{cr}(a, b, c, d) = \mathrm{cr}(c, d, a, b)$.

Example. $\mathrm{cr}(\infty, -1, 0, \lambda) = \lambda$ for $\lambda \in (0, \infty)$.

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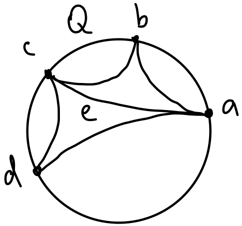
Definition. The *shear* of a quadrilateral Q with vertices $a, b, c, d \in \mathbb{T}$ along its diagonal $e = (a, c)$ is defined



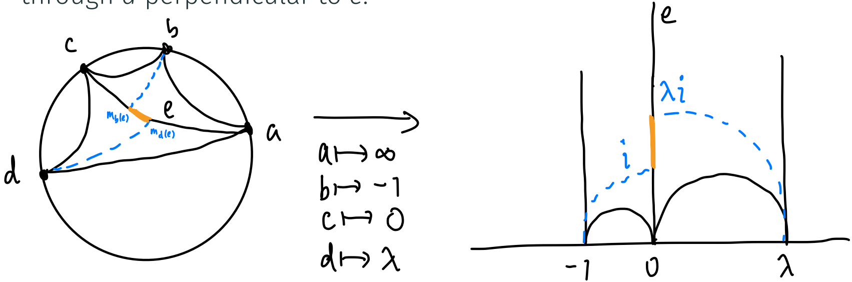
$$s(Q, e) = \log \mathrm{cr}(a, b, c, d).$$

Shear in terms of hyperbolic length

The shear $s(Q, e)$ can also be computed as (signed) hyperbolic length:

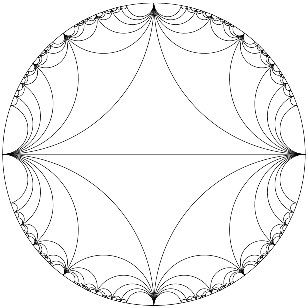
$$s(Q, e) = \pm \mathrm{d}_{\mathrm{hyp}}(m_b(e), m_d(e)).$$

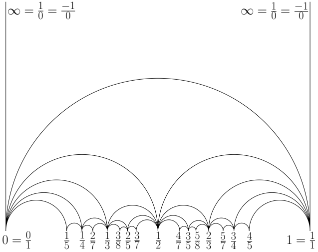
Here $m_b(e)$ is the intersection of the geodesic through b perpendicular to e and $m_d(e)$ is the intersection of the geodesic through d perpendicular to e.



The shear $s(Q, e)$ measures how the two triangles on either side of e are glued together to construct Q.

Farey tessellation





$\mathcal{F} = (V, E)$ tessellation of $\mathbb{D}$ starting from $\tau_0 = \{-1, \mathfrak{i}, 1\}$ generated by hyperbolic reflections. Vertices $V = \mathbb{T} \cap \mathbb{Q}^2$. The *dual tree* $\mathcal{F}^*$ (where each triangle corresponds to a vertex, etc) is a trivalent tree.

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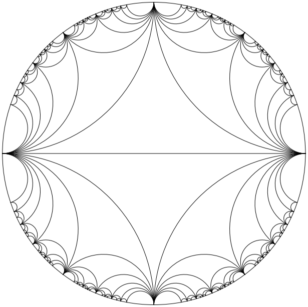

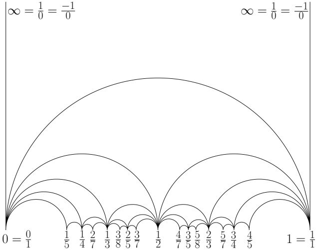

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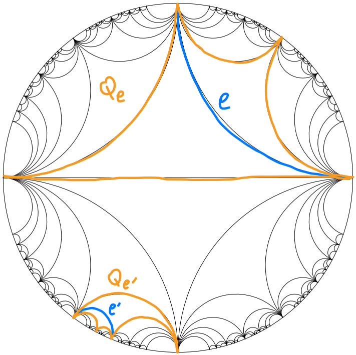
Conjugating to $\mathbb{H}$ by a Möbius transformation sending $\{-1, \mathfrak{i}, 1\} \mapsto \{0, 1, \infty\}$, V is sent to $\mathbb{Q}$. There is an edge $(p/q, r/s) \in E$ if and only if $pr - qs = 1$, and tessellation is invariant under the action of $\mathrm{PSL}(2, \mathbb{Z})$ action.

Farey tessellation in $\mathbb{D}$ and shears

The *Farey quad* Q_e around $e \in E$ is the pair of triangles in $\mathcal{F}$ with diagonal e.



Since $\mathcal{F}$ is generated by reflection, in terms of shears

$$s(Q_e, e) = 0 \qquad \forall\, e \in E.$$

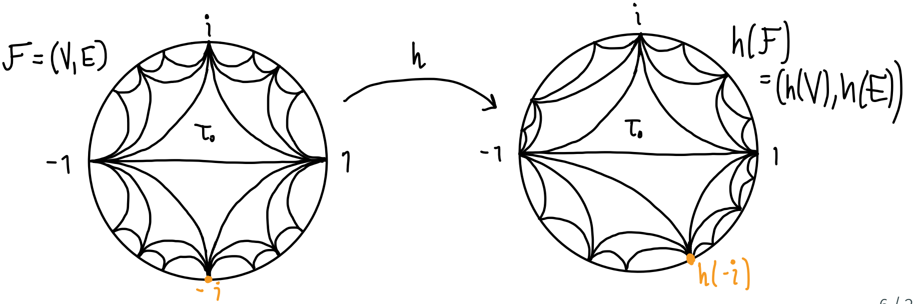
Tessellation from homeomorphism

A Möbius transformations is determined by its action on three points, so

$$\mathrm{Homeo}^+(\mathbb{T})/\mathrm{Möb} \cong \{h \in \mathrm{Homeo}^+(\mathbb{T}) : h \text{ fixes } \pm 1, \mathfrak{i}\}.$$

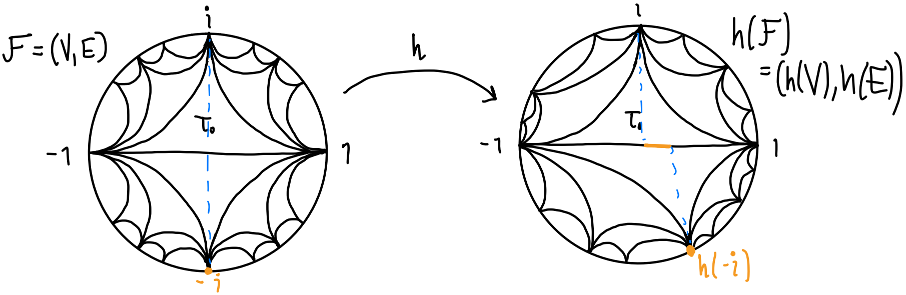
Given a homeomorphism h fixing $\pm 1, \mathfrak{i}$, we can define a tessellation $h(\mathcal{F})$ which contains the triangle τ_0 and has vertices $h(V)$ and edges

$$h(E) = \{(h(a), h(b)) : (a, b) \in E\}.$$



Definition. If h is a homeomorphism, its *shear coordinate* $s_h : E \to \mathbb{R}$ is

$$s_h(e) = s(h(Q_e), h(e)) \qquad \forall\, e \in E.$$



Remark. Not all functions $s : E \to \mathbb{R}$ encode homeomorphisms. (There exist shear functions where the image of V is not dense.)

Circle homeomorphisms

Quasisymmetric homemorphisms QS($\mathbb{T}$)

A map $f : \mathbb{D} \to \mathbb{D}$ is *quasiconformal* if f solves the *Beltrami equation*

$$f_{\bar{z}} = \mu f_z,$$

for some *Beltrami coefficient* μ with $||\mu||_\infty < 1$.

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Shears for quasisymmetric maps are totally classified (Saric). One model of *universal Teichmüller space* is QS($\mathbb{T}$)/Möb, where Möb $=$ PSU$(1,1)$ is the Möbius transformations preserving the disk.

Weil-Petersson homeomorphisms $\mathrm{WP}(\mathbb{T})$

Weil-Petersson Teichmüller space $\mathrm{WP}(\mathbb{T})/\mathrm{M\ddot{o}b}$ is a subspace of universal Teichmüller space that has received a lot of interest lately. The class now has at least 26 definitions, often with L^2 structure.

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Definition 1. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is Weil-Petersson if *there exists* an extension $f : \mathbb{D} \to \mathbb{D}$ such that:

- f is quasiconformal, i.e. f solves $f_{\bar{z}} = \mu f_z$ for μ with $||\mu||_\infty < 1$.
- The Beltrami coefficient μ is in L^2 for the hyperbolic metric on the disk, i.e.

$$\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1-|z|^2)^2} \, \mathrm{d}A(z) < \infty.$$

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Definition 2 (Shen). A homoemorphism h is Weil-Petersson if and only if it is absolutely continuous and $\log h' \in H^{1/2}$, i.e.

$$\iint_{\mathbb{T}\times\mathbb{T}} \left| \frac{\log h'(x) - \log h'(y)}{x - y} \right|^2 \mathrm{d}x\mathrm{d}y < \infty.$$

Weil-Petersson and Hölder classes

We define *Hölder classes*

$$\mathcal{C}^{1,\alpha} = \{h : \mathbb{T} \to \mathbb{T} : \log h' \text{ is } \alpha\text{-Hölder}\}.$$

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If $\log h'$ is α-Hölder, then

$$\iint_{\mathbb{T}\times\mathbb{T}} \left| \frac{\log h'(x) - \log h'(y)}{x - y} \right|^2 \mathrm{d}x\mathrm{d}y \leq \text{const.} \iint_{\mathbb{T}\times\mathbb{T}} |x-y|^{2\alpha-2}\, \mathrm{d}x\mathrm{d}y,$$

which is finite if and only if $2\alpha - 2 > -1$ hence if and only if $\alpha > 1/2$.

Square summable shears

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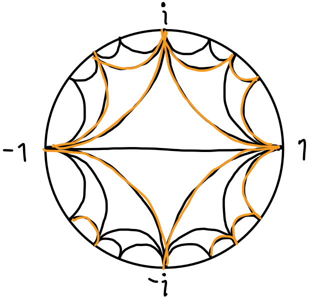
This will motivate the definition of *diamond shear* coordinates, and the space $\mathcal{H}$ (square summable diamond shears), which we show is much closer to $\mathrm{WP}(\mathbb{T})$.

Circle homeomorphisms with finitely supported shears

Finitely supported is piecewise Möbius

A shear function $s : E \to \mathbb{R}$ is *finitely supported* if $s(e) \neq 0$ for only finitely many $e \in E$.

Example



edges e in the support of s.

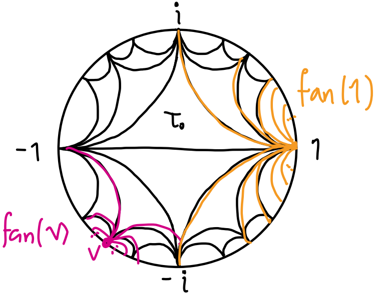
A finitely supported shear function is always induced by a piecewise Möbius circle homeomorphism with “breakpoints” in the vertices V of $\mathcal{F}$.

For $v \in V$, fan(v) is the edges $e \in E$ incident to v.

Lemma. If $h : \mathbb{T} \to \mathbb{T}$ has $s_h : E \to \mathbb{R}$ finitely supported, TFAE:

1. h is Weil Petersson;
2. h is $\mathcal{C}^{1,1}$ with breakpoints in V;
3. The shear function s_h satisfies the *finite balance condition*, i.e. for all $v \in V$, $\sum_{e \in \text{fan}(v)} s(e) = 0$.

Note: The shear function $s : E \to \mathbb{R}$ supported on one edge is *not* WP.

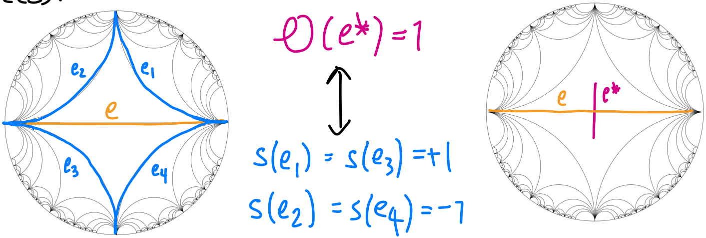


Definition of diamond shear

Definition. Fix $e \in E$ (with dual edge e^*), and let e_1, e_2, e_3, e_4 be the edges around Q_e. The *diamond shear* basis element Δ_e corresponds to the shear function with $s(e_1) = s(e_3) = +1$, $s(e_2) = s(e_4) = -1$, and all other shears 0.

Δ_e is the shear coordinate of a piecewise Möbius map with 4 pieces.

Ex Δ_e for $e = (-1, 1)$



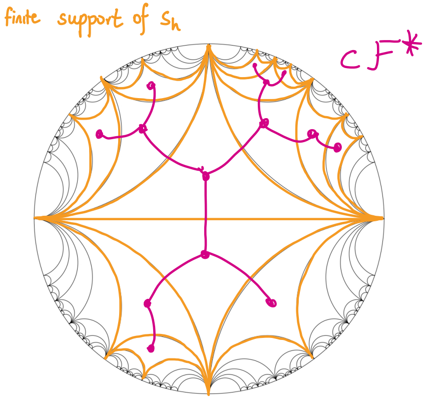
Definition. If a homeomorphism h has a shear coordinate $s : E \to \mathbb{R}$ such that $s = \sum_{e^* \in E^*} \vartheta(e^*)\, \Delta_e$ then h has *diamond shear coordinate* $\vartheta : E^* \to \mathbb{R}$.

Note: not all shear functions can be written as diamond shears.

Finitely supported diamond shears and Weil-Petersson

Lemma. If h has s_h finitely supported, then h is Weil-Petersson if and only if h has a diamond shear coordinate.

Proof sketch. “Pruning the tree.” By the previous Lemma, h is Weil-Petersson if and only if s_h satisfies the finite balanced condition.



Definition of $\mathcal{H}$

Not all shear functions can be written as diamond shears. We let $\mathcal{P} \subset \mathbb{R}^E$ be the subset of shear functions s such that s can be written in terms of diamond shear coordinates ϑ, $\Psi : \mathcal{P} \to \mathbb{R}^{\mathbb{E}^*}$ sends $s \mapsto \vartheta$.

Definition. The set of *square summable diamond shears* is

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Remark. From the condition for a shear function to encode a quasisymmetric homeomorphism, it follows that

$$\mathcal{H} \subset \mathrm{QS}(\mathbb{T}).$$

In particular, all $s \in \mathcal{H}$ induce homeomorphisms.

Main theorems

Theorem (SWW).

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Theorem (SWW). Suppose that $h, (h_n)_{n\geq 1} \in \mathcal{H}$ with diamond shear coordinates ϑ, ϑ_n respectively. If

$$\lim_{n\to\infty} \sum_{e^* \in E^*} (\vartheta_n(e^*) - \vartheta(e^*))^2 = 0$$

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Corollary. Piecewise-Möbius, C^1 maps with rational breakpoints are dense in $\mathcal{H}$ and $\mathrm{WP}(\mathbb{T})$.

Proof ideas

Proof ideas: $\mathcal{H} \subset \mathrm{WP}$

Given $h \in \mathcal{H}$, we explicitly construct an extension $f : \mathbb{D} \to \mathbb{D}$ and show it is quasiconformal and has Beltrami coefficient $\mu \in L^2(\mathbb{D}, \mathrm{d}_{\text{hyp}})$.

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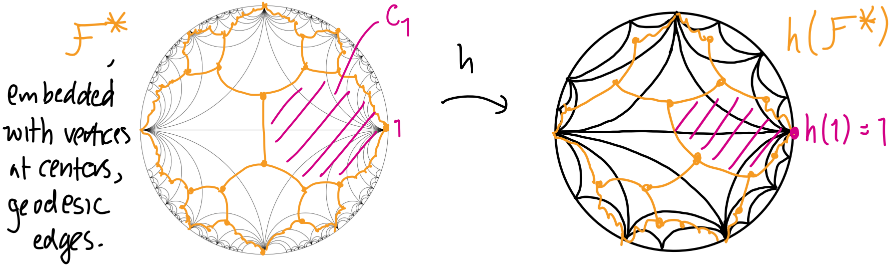
The dual tree $\mathcal{F}^*$ subdivides $\mathbb{D}$ into *cells* $\{C_v : v \in V\}$. Based on a construction by Kahn and Markovic, we construct f extending h that sends cells to cells.



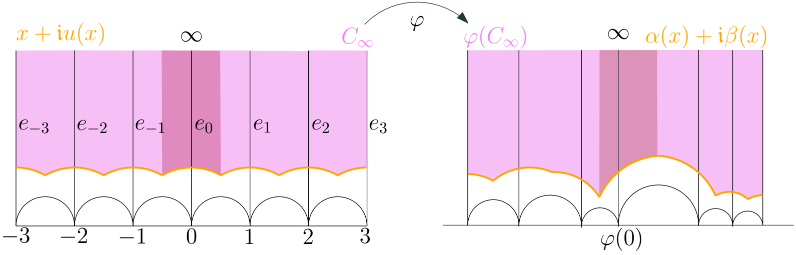
Step 1. Extend over $\mathcal{F}^*$ by hyperbolic stretching.

Step 2. Extend over a single cell C_v, $v \in V$.

Step 3. Stitch cells together again.

Proof ideas: extending over a cell C_v

For any $v \in V$, conjugating by appropriate Möbius transformations $\mathbb{H} \to \mathbb{D}$, we can send $h : \mathbb{T} \to \mathbb{T}$ to $\varphi : \mathbb{R} \to \mathbb{R}$ fixing ∞ and C_v to C_∞. Suffices to explain how to extend over C_∞.



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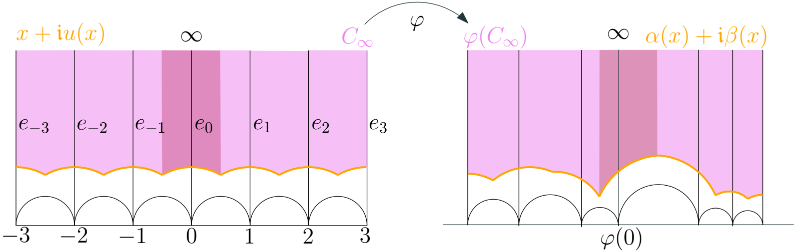

Extension ψ (conjugate to f) of φ sends $x + \mathrm{i}u(x)$ (boundary of C_∞) by hyperbolic stretching to the curve $\alpha(x) + \mathrm{i}\beta(x)$. We extend over the rest of the cell on vertical lines:

$$\psi(x + \mathrm{i}y) = \alpha(x) + \mathrm{i}(\beta(x) - u(x) + y) \qquad x + \mathrm{i}y \in C_\infty.$$

Proof ideas: $\mathcal{C}^{1,\alpha} \subset \mathcal{H}$

Analytic definition of diamond shears. If h has a diamond shear coordinate ϑ_h and $e = (a, b) \in E$, then

$$\vartheta_h(e) = \frac{1}{2} \log h'(a)h'(b) - \log \frac{h(a) - h(b)}{a - b}.$$

Summability of Farey lengths. Let $\ell(a, b)$ be the length of the shorter circular arc from a to b.

$$\sum_{(a,b)\in E} \ell(a, b)^r < \infty$$

if and only if $r > 1$.

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$$\vartheta_h(e) = \frac{1}{2} \log h'(a)h'(b) - \log \frac{h(a) - h(b)}{a - b}.$$

Summability of Farey lengths. Let $\ell(a, b)$ be the length of the shorter circular arc from a to b.

$$\sum_{(a,b)\in E} \ell(a, b)^r < \infty$$

if and only if $r > 1$.

Proof sketch. Suppose $h \in \mathcal{C}^{1,\alpha}$. By the mean value theorem*, there is $c \in (a, b)$ so that

$$\vartheta_h(e) = \frac{1}{2}(\log h'(a) - \log h'(c)) + \frac{1}{2}(\log h'(b) - \log h'(c)).$$

Since $\log h'$ is α-Hölder, $|\vartheta_h(e)|^2 \leq \text{const.}\, \ell(a, b)^{2\alpha}$, and the right hand side is summable if and only if $\alpha > 1/2$.

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Remark. However one can compute that for $n > 1$,

$$s_\varphi((n, \infty)) = \frac{1}{n \log n} + O\left(\frac{1}{n^2}\right).$$

This *is* square summable, corresponding to the fact that $\mathrm{WP}(\mathbb{T}) \subset \mathcal{S}$. The proof that $\mathrm{WP}(\mathbb{T}) \subset \mathcal{S}$ uses a necessary condition for WP due to C. Wu.

Thank you for listening!